

Calculus III: Coursework 6
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The deadline is 4PM on Tuesday Nov 20th. Hand-in is to the green box on the ground floor of the Maths building.

Note: there are more questions than usual but three of them are very short if done the right way.

1. Find, to the nearest degree, the Earth polar coordinates of the following cities: [8]

- (a) New York City (USA)
- (b) Adelaide (Australia)

Answer

Upon looking up latitudes and longitudes:

- (a) New York is at 41N 74W, so $\theta = 49^\circ$, $\phi = 286^\circ$.
- (b) Adelaide is at 35S 139E, so $\theta = 125^\circ$, $\phi = 139^\circ$

Marks: Having the concept clear (4), lookups and calculation (2×2) [Marker: allow 1 of the 2 marks if you get $\theta = 50$ for New York, and $\phi = 138$ for Adelaide; I do not want to penalize students too hard for failing to remember that there are 60 minutes in a degree rather than 100 (though if they use Google Map they will get the answer in decimals of a degree, anyway: it's Wikipedia that uses minutes). The more detailed figures are respectively $40^\circ 43' \text{N}$, and $138^\circ 35' \text{W}$ respectively.]

2. Invert the relation [5]

$$\begin{pmatrix} \mathbf{e}_r \\ \mathbf{e}_\theta \\ \mathbf{e}_\phi \end{pmatrix} = \begin{pmatrix} \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ -\sin \phi & \cos \phi & 0 \end{pmatrix} \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{pmatrix}.$$

for spherical polar coordinates, to give $\mathbf{i}, \mathbf{j}, \mathbf{k}$ in terms of $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi$.

Answer

Since the matrix is a rotation matrix, the inverse matrix is just the transpose, so

$$\begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{pmatrix} = \begin{pmatrix} \sin \theta \cos \phi & \cos \theta \cos \phi & -\sin \phi \\ \sin \theta \sin \phi & \cos \theta \sin \phi & \cos \phi \\ \cos \theta & -\sin \theta & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}_r \\ \mathbf{e}_\theta \\ \mathbf{e}_\phi \end{pmatrix}.$$

Marks: 3 for a correct method, 2 for accuracy (students may try to evaluate the determinant and cofactors).

3. Show that if position $\mathbf{r}$ is dependent on time t then in spherical polar coordinates [20]

$$\dot{\mathbf{e}}_r = \dot{\theta}\mathbf{e}_\theta + \dot{\phi}\sin\theta\mathbf{e}_\phi, \quad \dot{\mathbf{e}}_\theta = -\dot{\theta}\mathbf{e}_r + \dot{\phi}\cos\theta\mathbf{e}_\phi, \quad \dot{\mathbf{e}}_\phi = -\dot{\phi}(\sin\theta\mathbf{e}_r + \cos\theta\mathbf{e}_\theta).$$

where the dot as usual means the derivative with respect to time.

Hence show that the velocity and acceleration are

$$\begin{aligned}\dot{\mathbf{r}} &= \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta + r\sin\theta\dot{\phi}\mathbf{e}_\phi, \\ \ddot{\mathbf{r}} &= (\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\sin\theta\cos\theta\dot{\phi}^2)\mathbf{e}_\theta \\ &\quad + (r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta)\mathbf{e}_\phi.\end{aligned}$$

Answer

$$\begin{aligned}\mathbf{e}_r &= \sin\theta\cos\phi\mathbf{i} + \sin\theta\sin\phi\mathbf{j} + \cos\theta\mathbf{k} \text{ so} \\ \dot{\mathbf{e}}_r &= \dot{\theta}(\cos\theta\cos\phi\mathbf{i} + \cos\theta\sin\phi\mathbf{j} - \sin\theta\mathbf{k}) + \dot{\phi}(-\sin\theta\sin\phi\mathbf{i} + \sin\theta\cos\phi\mathbf{j}) \\ &= \dot{\theta}\mathbf{e}_\theta + \dot{\phi}\sin\theta\mathbf{e}_\phi.\end{aligned}$$

Similarly

$$\begin{aligned}\mathbf{e}_\theta &= \cos\theta\cos\phi\mathbf{i} + \cos\theta\sin\phi\mathbf{j} - \sin\theta\mathbf{k} \text{ so} \\ \dot{\mathbf{e}}_\theta &= \dot{\theta}(-\sin\theta\cos\phi\mathbf{i} - \sin\theta\sin\phi\mathbf{j} - \cos\theta\mathbf{k}) + \dot{\phi}(-\cos\theta\sin\phi\mathbf{i} + \cos\theta\cos\phi\mathbf{j}) \\ &= -\dot{\theta}\mathbf{e}_r + \dot{\phi}\cos\theta\mathbf{e}_\phi. \\ \mathbf{e}_\phi &= -\sin\phi\mathbf{i} + \cos\phi\mathbf{j} \text{ so} \\ \dot{\mathbf{e}}_\phi &= \dot{\phi}(-\cos\phi\mathbf{i} - \sin\phi\mathbf{j}) \\ &= \dot{\phi}(-\cos\phi(\sin\theta\cos\phi\mathbf{e}_r + \cos\theta\cos\phi\mathbf{e}_\theta - \sin\phi\mathbf{e}_\phi) \\ &\quad - \sin\phi(\sin\theta\sin\phi\mathbf{e}_r + \cos\theta\sin\phi\mathbf{e}_\theta + \cos\phi\mathbf{e}_\phi)) \\ &= -\dot{\phi}(\sin\theta\mathbf{e}_r + \cos\theta\mathbf{e}_\theta).\end{aligned}$$

Using these results and $\mathbf{r} = r\mathbf{e}_r$ we get

$$\begin{aligned}\dot{\mathbf{r}} &= \dot{r}\mathbf{e}_r + r\dot{\mathbf{e}}_r = \dot{r}\mathbf{e}_r + r(\dot{\theta}\mathbf{e}_\theta + \dot{\phi}\sin\theta\mathbf{e}_\phi). \\ \ddot{\mathbf{r}} &= \ddot{r}\mathbf{e}_r + 2\dot{r}\dot{\mathbf{e}}_r + r\ddot{\mathbf{e}}_r \text{ and, differentiating } \dot{\mathbf{e}}_r \text{ again,} \\ \ddot{\mathbf{e}}_r &= \ddot{\theta}\mathbf{e}_\theta + \dot{\theta}\dot{\mathbf{e}}_\theta + \ddot{\phi}\sin\theta\mathbf{e}_\phi + \dot{\phi}\dot{\theta}\cos\theta\mathbf{e}_\phi + \dot{\phi}\sin\theta\dot{\mathbf{e}}_\phi \\ &= \ddot{\theta}\mathbf{e}_\theta + \dot{\theta}(-\dot{\theta}\mathbf{e}_r + \dot{\phi}\cos\theta\mathbf{e}_\phi) + \ddot{\phi}\sin\theta\mathbf{e}_\phi + \dot{\phi}\dot{\theta}\cos\theta\mathbf{e}_\phi \\ &\quad + \dot{\phi}\sin\theta(-\dot{\phi}(\sin\theta\mathbf{e}_r + \cos\theta\mathbf{e}_\theta)) \\ &= -(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta)\mathbf{e}_r + (\ddot{\theta} - \dot{\phi}^2\cos\theta\sin\theta)\mathbf{e}_\theta + (\ddot{\phi}\sin\theta + 2\dot{\phi}\dot{\theta}\cos\theta)\mathbf{e}_\phi\end{aligned}$$

and the result now comes from substituting back $\dot{\mathbf{e}}_r$ and $\ddot{\mathbf{e}}_r$ in the expression for $\ddot{\mathbf{r}}$.

Marks: 3 for a correct method for the derivatives of the unit vectors, 2 for each accurate answer. 2 for $\dot{\mathbf{e}}_r$. 3 for method for $\ddot{\mathbf{r}}$, 4 for correct $\ddot{\mathbf{e}}_r$ and 2 for final answer for acceleration.

4. Show that if $x = u^2 - v^2$, $y = 2uv$, then the coordinates u and v are orthogonal. [6]

Answer

$\mathbf{r}_u = 2u\mathbf{i} + 2v\mathbf{j}$, $\mathbf{r}_v = -2v\mathbf{i} + 2u\mathbf{j}$, so $\mathbf{r}_u \cdot \mathbf{r}_v = 0$, whence they are orthogonal.

Marks: 3 for understanding what needs to be done, 3 for accuracy.

5. In cylindrical polar coordinates (ρ, ϕ, z) , evaluate [12]

(a) ∇f

(b) $\nabla \cdot \mathbf{F}$

(c) $\nabla \times \mathbf{F}$

where $f = \rho z^2 \cos \phi$ and $\mathbf{F} = (\rho^2 - \sin \phi)\mathbf{e}_\rho + \cos \phi \mathbf{e}_\phi$. (You may ignore the problem of well-definedness of values at $\rho = 0$.)

Answer

[This and the next question just require substitution into the formulae collected at the end of chapter 5 in the online notes. Example 5.7 in the notes helps with part (b) here.]

(a)

$$\nabla f = \frac{\partial f}{\partial \rho} \mathbf{e}_\rho + \frac{1}{\rho} \frac{\partial f}{\partial \phi} \mathbf{e}_\phi + \frac{\partial f}{\partial z} \mathbf{e}_z = z^2 \cos \phi \mathbf{e}_\rho - z^2 \sin \phi \mathbf{e}_\phi + 2\rho z \cos \phi \mathbf{e}_z.$$

(b) Using the example,

$$\nabla \cdot \mathbf{F} = \frac{1}{\rho} \left[\frac{\partial(\rho^3 - \rho \sin \phi)}{\partial \rho} + \frac{\partial \cos \phi}{\partial \phi} \right] = \frac{3\rho^2 - 2 \sin \phi}{\rho}.$$

(c)

$$\nabla \times \mathbf{F} = \frac{1}{\rho} \begin{vmatrix} \mathbf{e}_\rho & \rho \mathbf{e}_\phi & \mathbf{e}_z \\ \partial/\partial \rho & \partial/\partial \phi & \partial/\partial z \\ F_\rho & \rho F_\phi & F_z \end{vmatrix} = \frac{1}{\rho} \begin{vmatrix} \mathbf{e}_\rho & \rho \mathbf{e}_\phi & \mathbf{e}_z \\ \partial/\partial \rho & \partial/\partial \phi & \partial/\partial z \\ (\rho^2 - \sin \phi) & \rho \cos \phi & 0 \end{vmatrix} = \frac{2 \cos \phi}{\rho} \mathbf{e}_z.$$

Marks: 3 for the gradient, 4 for the divergence, 5 for the curl (no marks for any part where the answer should be a scalar and is given as a vector, or vice versa).

6. In spherical polar coordinates (r, θ, ϕ) , evaluate [12]

(a) ∇f

(b) $\nabla \cdot \mathbf{F}$

(c) $\nabla \times \mathbf{F}$

where $f = \sin \theta \sin \phi / r$ and $\mathbf{F} = r \cos \theta \mathbf{e}_\phi$. (You may ignore the problem of well-definedness of values at $r = 0$, $\theta = 0$ and $\theta = \pi$.)

Answer [Part (a) of this one is very like Example 5.6 in the notes, and part (c) follows from Example 5.8.] (a)

$$\begin{aligned}\nabla f &= \frac{\partial(\sin \theta \sin \phi / r)}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial(\sin \theta \sin \phi / r)}{\partial \theta} \mathbf{e}_\theta + \frac{1}{r \sin \theta} \frac{\partial(\sin \theta \sin \phi / r)}{\partial \phi} \mathbf{e}_\phi \\ &= (-\sin \theta \sin \phi \mathbf{e}_r + \cos \theta \sin \phi \mathbf{e}_\theta + \cos \phi \mathbf{e}_\phi) / r^2.\end{aligned}$$

(b)

$$\nabla \cdot \mathbf{F} = \frac{1}{r^2 \sin \theta} \left[\frac{\partial(r^2 \sin \theta F_r)}{\partial r} + \frac{\partial(r \sin \theta F_\theta)}{\partial \theta} + \frac{\partial(r F_\phi)}{\partial \phi} \right] = \frac{1}{r^2 \sin \theta} \frac{\partial(r^2 \cos \theta)}{\partial \phi} = 0.$$

(c) Using the result of the example mentioned above,

$$\begin{aligned}\nabla \times \mathbf{F} &= \frac{1}{r^2 \sin \theta} \left[\frac{\partial(r^2 \sin \theta \cos \theta)}{\partial \theta} \right] \mathbf{e}_r + \frac{1}{r \sin \theta} \left[-\frac{\partial(r^2 \sin \theta \cos \theta)}{\partial r} \right] \mathbf{e}_\theta \\ &= \frac{\cos^2 \theta - \sin^2 \theta}{\sin \theta} \mathbf{e}_r - 2 \cos \theta \mathbf{e}_\theta.\end{aligned}$$

Marks: 3 for the gradient, 4 for the divergence, 5 for the curl (no marks for any part where the answer should be a scalar and is given as a vector, or vice versa).

7. A cylinder is described in cylindrical coordinates (ρ, ϕ, z) by $\rho \leq 2$, $0 \leq z \leq 3$. [17]
Let

$$\mathbf{F} = \rho z \sin \phi (\cos \phi \mathbf{e}_\rho - \sin \phi \mathbf{e}_\phi + \cos \phi \mathbf{e}_z).$$

Find $\nabla \times \mathbf{F}$ and evaluate the integral $\int_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$ by direct integration over the surface S consisting of the base ($z = 0$) and the curved side of the cylinder, with normal taken downwards on the base.

Answer

$$\begin{aligned}\nabla \times \mathbf{F} &= \frac{1}{\rho} \begin{vmatrix} \mathbf{e}_\rho & \rho \mathbf{e}_\phi & \mathbf{e}_z \\ \partial/\partial \rho & \partial/\partial \phi & \partial/\partial z \\ F_\rho & \rho F_\phi & F_z \end{vmatrix} = \frac{1}{\rho} \begin{vmatrix} \mathbf{e}_\rho & \rho \mathbf{e}_\phi & \mathbf{e}_z \\ \partial/\partial \rho & \partial/\partial \phi & \partial/\partial z \\ \rho z \sin \phi \cos \phi & -\rho^2 z \sin^2 \phi & \rho z \cos \phi \sin \phi \end{vmatrix} \\ &= \frac{1}{\rho} \left[\mathbf{e}_\rho (\rho z (\cos^2 \phi - \sin^2 \phi)) + \rho \mathbf{e}_\phi (\rho \cos \phi \sin \phi) + \mathbf{e}_z (-2 \rho z \sin^2 \phi) \right. \\ &\quad \left. - \mathbf{e}_\rho (-\rho^2 \sin^2 \phi) - \rho \mathbf{e}_\phi (z \cos \phi \sin \phi) - \mathbf{e}_z \rho z (\cos^2 \phi - \sin^2 \phi) \right] \\ &= (z \cos 2\phi + \rho \sin^2 \phi) \mathbf{e}_\rho + (\rho - z) \cos \phi \sin \phi \mathbf{e}_\phi - z \mathbf{e}_z.\end{aligned}$$

On the base, $z = 0$ and $\mathbf{n} = -\mathbf{e}_z$ so $\nabla \times \mathbf{F} \cdot d\mathbf{S} = 0$. On the curved surface $d\mathbf{S} = \rho dz d\phi \mathbf{e}_\rho$ and $\rho = 2$ so $\nabla \times \mathbf{F} \cdot d\mathbf{S} = (z \cos 2\phi + \rho \sin^2 \phi) \rho dz d\phi = (2z \cos 2\phi +$

$4 \sin^2 \phi) dz d\phi$ and the integral becomes

$$\begin{aligned} \int_0^{2\pi} \int_0^3 [2z \cos 2\phi + 2(1 - \cos 2\phi)] dz d\phi &= [z^2]_0^3 [\tfrac{1}{2} \sin 2\phi]_0^{2\pi} + [2z]_0^3 [1 - \tfrac{1}{2} \sin 2\phi]_0^{2\pi} \\ &= 12\pi \end{aligned}$$

Marks: 6 for the curl (messier than previous example). 3 for integral over the base. 4 for forming correct integral over the curved surface, 4 for accuracy in evaluation.

8. [From the 2002 exam] [20]
In spherical polar coordinates the vector field $\mathbf{F}$ is given by

$$\mathbf{F} = r \sin \theta (\mathbf{e}_r + \mathbf{e}_\theta + \mathbf{e}_\phi).$$

Verify the Divergence Theorem for this vector field over a sphere of radius a centered at the origin.

Answer

$$\begin{aligned} \nabla \cdot \mathbf{F} &= \frac{1}{r^2 \sin \theta} \left[\frac{\partial(r^2 \sin \theta)(r \sin \theta)}{\partial r} + \frac{\partial(r \sin \theta)(r \sin \theta)}{\partial \theta} + \frac{\partial(r(r \sin \theta))}{\partial \phi} \right] \\ &= \frac{1}{r^2 \sin \theta} [3r^2 \sin^2 \theta + 2r^2 \sin \theta \cos \theta] \\ &= 3 \sin \theta + 2 \cos \theta \end{aligned}$$

The volume integral is

$$\begin{aligned} \int \nabla \cdot \mathbf{F} dV &= \int_0^{2\pi} \int_0^\pi \int_0^a (3 \sin \theta + 2 \cos \theta) r^2 \sin \theta dr d\theta d\phi \\ &= [\phi]_0^{2\pi} [r^3/3]_0^a \int_0^\pi (3 \sin^2 \theta + 2 \sin \theta \cos \theta) d\theta \\ &= \frac{2\pi a^3}{3} [\tfrac{3}{2}(\theta - \tfrac{1}{2} \sin 2\theta) + \sin^2 \theta]_0^\pi \\ &= \pi^2 a^3, \end{aligned}$$

using $\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$ in the integration.

Since on the surface of the sphere $d\mathbf{S} = r^2 \sin \theta d\theta d\phi \mathbf{e}_r = a^2 \sin \theta d\theta d\phi \mathbf{e}_r$, the surface integral becomes

$$\begin{aligned} \int \mathbf{F} \cdot d\mathbf{S} &= \int_0^{2\pi} \int_0^\pi (a \sin \theta) a^2 \sin \theta d\theta d\phi \\ &= a^3 [\phi]_0^{2\pi} \int_0^\pi \sin^2 \theta d\theta \\ &= 2\pi a^3 [\tfrac{1}{2}(\theta - \tfrac{1}{2} \sin 2\theta)]_0^\pi \\ &= \pi^2 a^3. \end{aligned}$$

Marks: 5 for the divergence. 3 for forming correct volume integral, 4 for evaluation. 4 for forming correct surface integral, 4 for evaluation.